Towards τ adaptivity for lithosphere dynamics

Non-smooth processes in heterogeneous media

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This talk: http://59A2.org/files/20140707-SIAMAnnual.pdf

Plan: ruthlessly eliminate communication

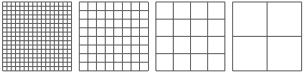
Why?

- Local recovery despite global coupling
- Tolerance for high-frequency load imbalance
  - From irregular computation or hardware error correction
- More scope for dynamic load balance

Requirements

- Must retain optimal convergence with good constants
- Flexible, robust, and debuggable

Multigrid Preliminaries



Multigrid is an $O(n)$ method for solving algebraic problems by defining a hierarchy of scale. A multigrid method is constructed from:

1. a series of discretizations
   - coarser approximations of the original problem
   - constructed algebraically or geometrically
2. intergrid transfer operators
   - residual restriction I_h^H (fine to coarse)
   - state restriction $\hat{I}_h^H$ (fine to coarse)
   - partial state interpolation I_H^h (coarse to fine, 'prolongation')
   - state reconstruction $\mathbb{I}_H^h$ (coarse to fine)
3. Smoothers (S)
   - correct the high frequency error components
   - Richardson, Jacobi, Gauss-Seidel, etc.
   - Gauss-Seidel-Newton or optimization methods

τ formulation of Full Approximation Scheme (FAS)

- classical formulation: "coarse grid *accelerates* fine grid $\searrow\nearrow$
- τ formulation: "fine grid feeds back into coarse grid" $\nearrow\searrow$
- To solve $Nu = f$, recursively apply

$$\text{pre-smooth} \qquad \tilde{u}^h \leftarrow S^h_{\text{pre}}(u^h_0, f^h)$$

$$\text{solve coarse problem for } u^H \qquad N^H u^H = \underbrace{I^H_h f^h}_{f^H} + \underbrace{N^H \hat{I}^H_h \tilde{u}^h - I^H_h N^h \tilde{u}^h}_{\tau^H_h}$$

$$\text{correction and post-smooth} \qquad u^h \leftarrow S^h_{\text{post}}\left(\tilde{u}^h + I^h_H(u^H - \hat{I}^H_h \tilde{u}^h), f^h\right)$$

I^H_h	residual restriction	$\hat{I}^H_h$	solution restriction
I^h_H	solution interpolation	$f^H = I^H_h f^h$	restricted forcing
$\{S^h_{\text{pre}}, S^h_{\text{post}}\}$	smoothing operations on the fine grid		

- At convergence, $u^{H*} = \hat{I}^H_h u^{h*}$ solves the τ-corrected coarse grid equation $N^H u^H = f^H + \tau^H_h$, thus τ^H_h is the "fine grid feedback" that makes the coarse grid equation accurate.
- τ^H_h is *local* and need only be recomputed where it becomes stale.
- Interpretation by Achi Brandt in 1977, many tricks followed

Model problem: $\mathfrak{p}$-Laplacian with slip boundary conditions

- 2-dimensional model problem for power-law fluid cross-section

$$-\nabla\cdot\left(|\nabla u|^{\mathfrak{p}-2}\nabla u\right) - f = 0, \qquad 1 \le \mathfrak{p} \le \infty$$

  Singular or degenerate when $\nabla u = 0$

- Regularized variant

$$-\nabla\cdot(\eta\nabla u) - f = 0$$

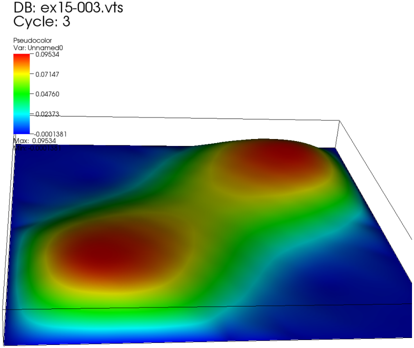
$$\eta(\gamma) = (\varepsilon^2 + \gamma)^{\frac{\mathfrak{p}-2}{2}} \qquad \gamma(u) = \frac{1}{2}|\nabla u|^2$$

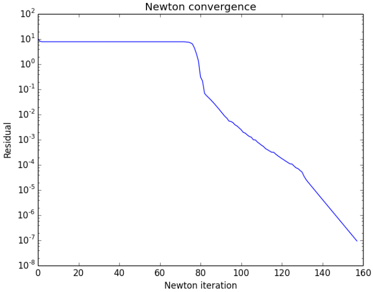
- Friction boundary condition on one side of domain

$$\nabla u\cdot n + A(x)\,|u|^{q-1}\,u = 0$$

Model problem: $\mathfrak{p}$-Laplacian with slip boundary conditions

- $\mathfrak{p} = 1.3$ and $q = 0.2$, checkerboard coefficients $\{10^{-2}, 1\}$
- Friction coefficient $A = 0$ in center, 1 at corners





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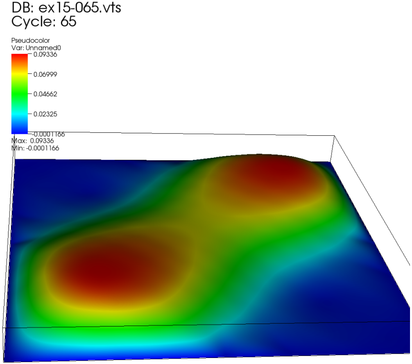

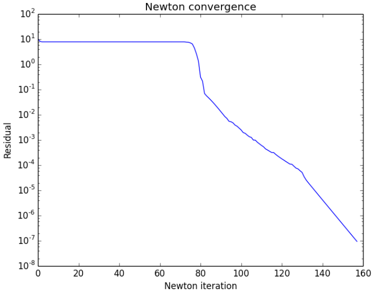

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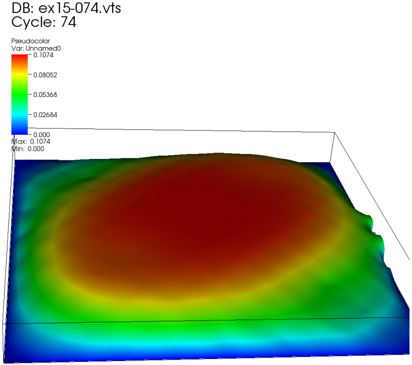

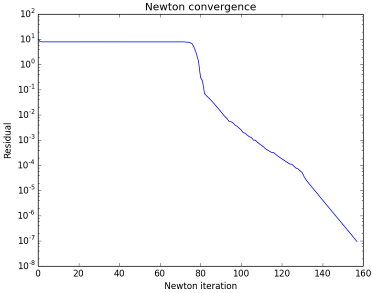

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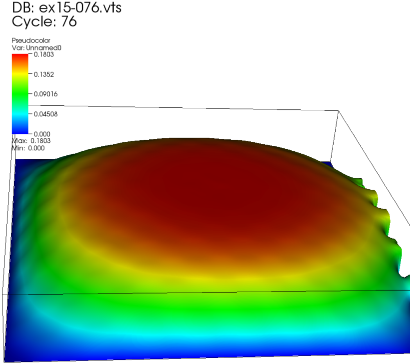

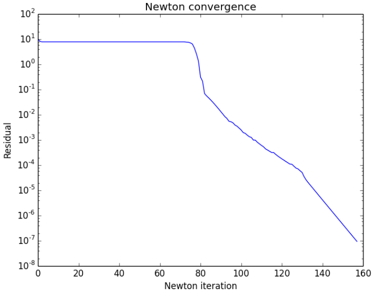

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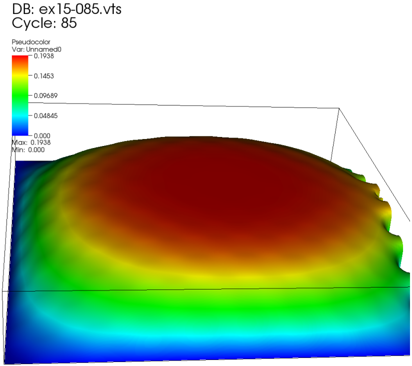

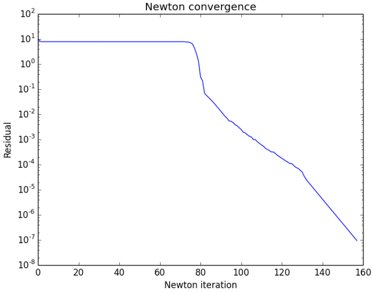

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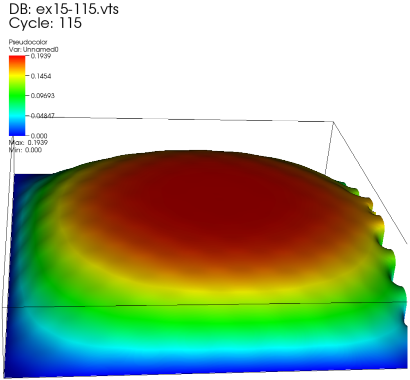
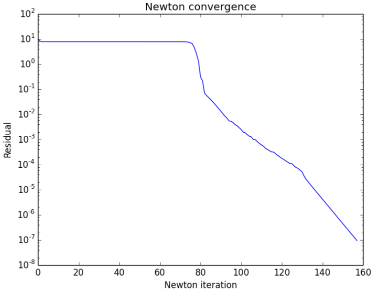
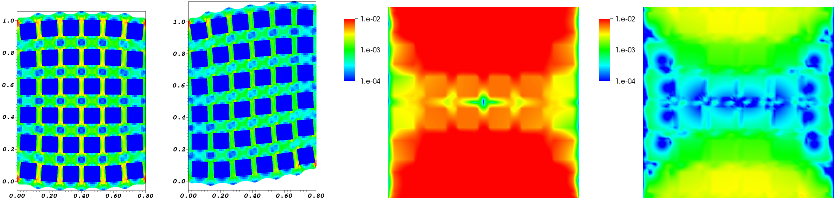




τ corrections



- Plane strain elasticity, $E = 1000, \nu = 0.4$ inclusions in $E = 1, \nu = 0.2$ material, coarsen by 3^2.
- Solve initial problem everywhere and compute $\tau_h^H = A^H \hat{I}_h^H u^h - I_h^H A^h u^h$
- Change boundary conditions and solve FAS coarse problem

$$N^H \acute{u}^H = \underbrace{I_h^H \acute{f}^h}_{\acute{f}^H} + \underbrace{N^H \hat{I}_h^H \tilde{u}^h - I_h^H N^h \tilde{u}^h}_{\tau_h^H}$$

- Prolong, post-smooth, compute error $e^h = \acute{u}^h - (N^h)^{-1} \acute{f}^h$
- Coarse grid *with* τ is nearly $10\times$ better accuracy

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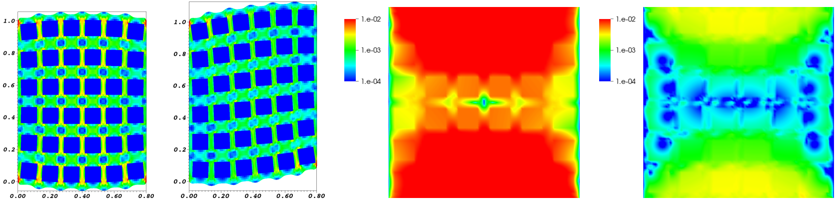

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τ adaptivity: an idea for heterogeneous media

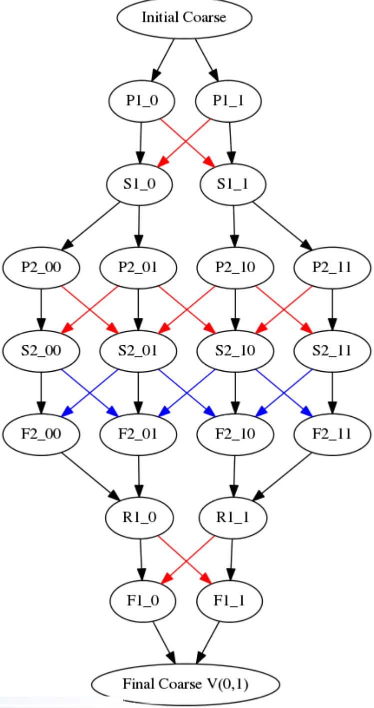
- Applications with localized nonlinearities
  - Subduction, rifting, rupture/fault dynamics
  - Carbon fiber, biological tissues, fracture
- Adaptive methods fail for heterogeneous media
  - Rocks are rough, solutions are not “smooth”
  - Cannot build accurate coarse space without scale separation
- τ adaptivity
  - Fine-grid work needed everywhere at first
  - Then τ becomes accurate in nearly-linear regions
  - Only visit fine grids in “interesting” places: active nonlinearity, drastic change of solution

Comparison to nonlinear domain decomposition

- ASPIN (Additive Schwarz preconditioned inexact Newton)
  - Cai and Keyes (2003)
  - More local iterations in strongly nonlinear regions
  - Each nonlinear iteration only propagates information locally
  - Many real nonlinearities are activated by long-range forces
    - locking in granular media (gravel, granola)
    - binding in steel fittings, crack propagation
  - Two-stage algorithm has different load balancing
    - Nonlinear subdomain solves
    - Global linear solve
- τ adaptivity
  - Minimum effort to communicate long-range information
  - Nonlinearity sees effects as accurate as with global fine-grid feedback
  - Fine-grid work always proportional to “interesting” changes

Low communication MG

- red arrows can be removed by τ-FAS with overlap
- blue arrows can also be removed, but then algebraic convergence stalls when discretization error is reached
- no simple way to check that discretization error is obtained
- if fine grid state is not stored, use compatible relaxation to complete prolongation P

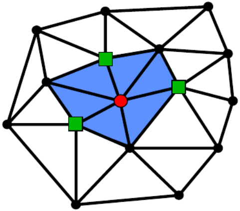


Nonlinear and matrix-free smoothing

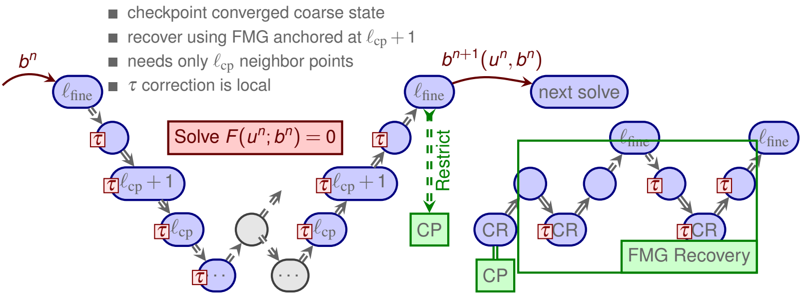
- matrix-based smoothers require global linearization
- nonlinearity often more efficiently resolved locally
- nonlinear additive or multiplicative Schwarz
- nonlinear/matrix-free is good if

$$C = \frac{(\text{cost to evaluate residual at one point}) \cdot N}{(\text{cost of global residual})} \sim 1$$

  - finite difference: $C < 2$
  - finite volume: $C \sim 2$, depends on reconstruction
  - finite element: $C \sim$ number of vertices per cell
- larger block smoothers help reduce C
- additive correction like Jacobi reduces C, but need to assemble corrector/scaling



Multiscale compression and recovery using τ form



- Normal multigrid cycles visit all levels moving from $n \rightarrow n+1$
- FMG recovery only accesses levels finer than ℓ_{CP}
- Lightweight checkpointing for transient adjoint computation
- Postprocessing applications, e.g., in-situ visualization at high temporal resolution in part of the domain

Outlook on τ-FAS adaptivity and compression

- Benefits of AMR without fine-scale smoothness
- Coarse-centric restructuring is a major interface change
- Nonlinear smoothers (and discretizations)
  - Smooth in neighborhood of “interesting” fine-scale features
  - Which discretizations can provide efficient matrix-free smoothers?
  - Does there exist an efficient smoother based on element Neumann problems?
- Dynamic load balancing
- Reliability of error estimates for refreshing τ
  - We want a coarse indicator for whether τ needs to change
- Worthwhile for resilience and to better use hardware