τ-adaptivity for nonsmooth processes in heterogeneous media

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Implicit solution of localized nonlinearities

Localized nonsmooth processes play a leading role in many geophysical problems, e.g.,

- plastic yielding, fracture
- frictional contact: faults, sub-glacial
- contact/collisions: marine glaciers, sedimentation
- phase change: ice/water/steam, magma

If the effects are primarily *local* (e.g., wetting and drying in coastal inundation), the nonsmoothness can be treated explicitly. But long-range stress transmission is instantaneous on the time scales of most geophysical problems, necessitating *implicit* treatment if time steps are to be chosen based on accuracy rather than stability.

Nonlinear solvers

The prevailing nonlinear solution algorithms are based on global linearization, using either Newton or Picard iteration.

$$F(u) = 0$$

$$\text{Solve:} \quad J(u)v = -F(u), \quad u \leftarrow u + v$$

$$\text{where} \quad J(u) \approx \nabla_u F(u)$$

- Each iteration requires a global linear solve (e.g., Krylov-Multigrid).
- Each iteration moves important information over large distances.
- Superlinear convergence not realized for nonsmooth problems.
- The number of iterations depends on the strength of the nonlinearity.

Model problem: $\mathfrak{p}$-Laplacian with friction

- 2-dimensional model problem for power-law fluid cross-section, $1 \le \mathfrak{p} \le \infty$

$$-\nabla \cdot (\eta \nabla u) - f = 0$$

$$\eta(\gamma) = \gamma_0(x)(\epsilon^2 + \gamma)^{\frac{\mathfrak{p}-2}{2}} \qquad \gamma(u) = \frac{1}{2}|\nabla u|^2$$

- Friction boundary condition, $0 \le q \le 1$

$$\nabla u \cdot \boldsymbol{n} + A(x)\,|u|^{q-1} u = 0$$

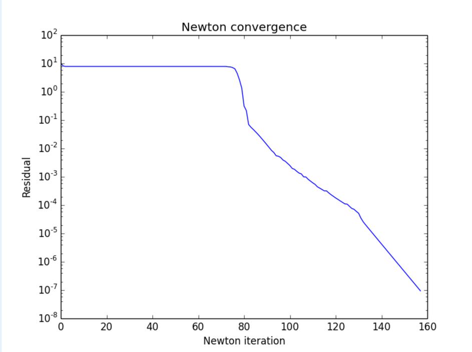
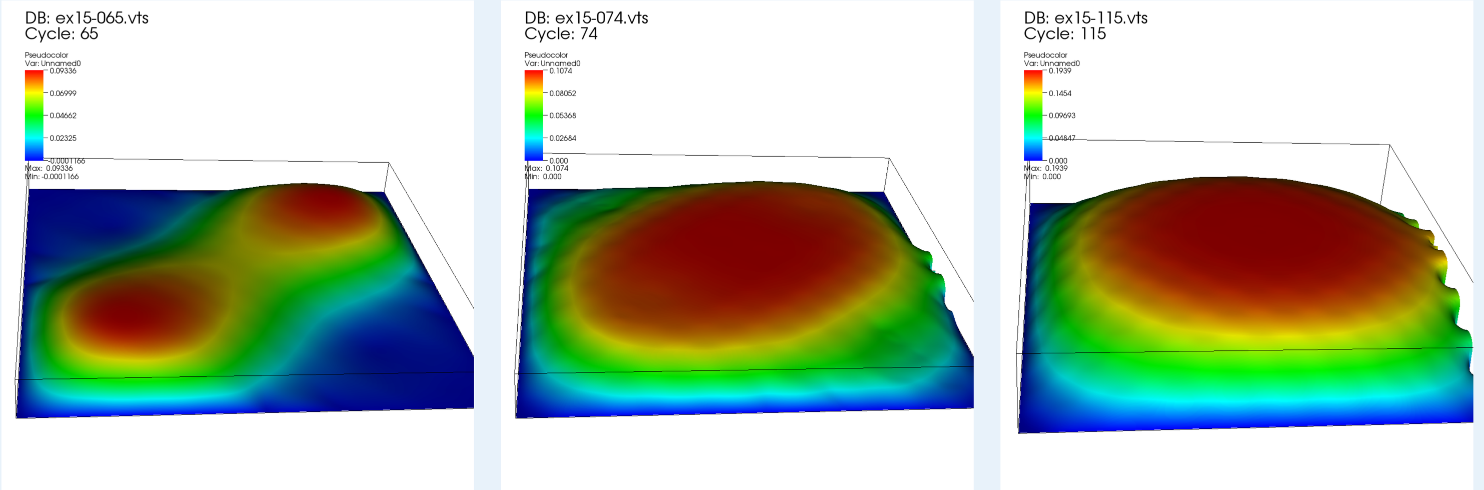


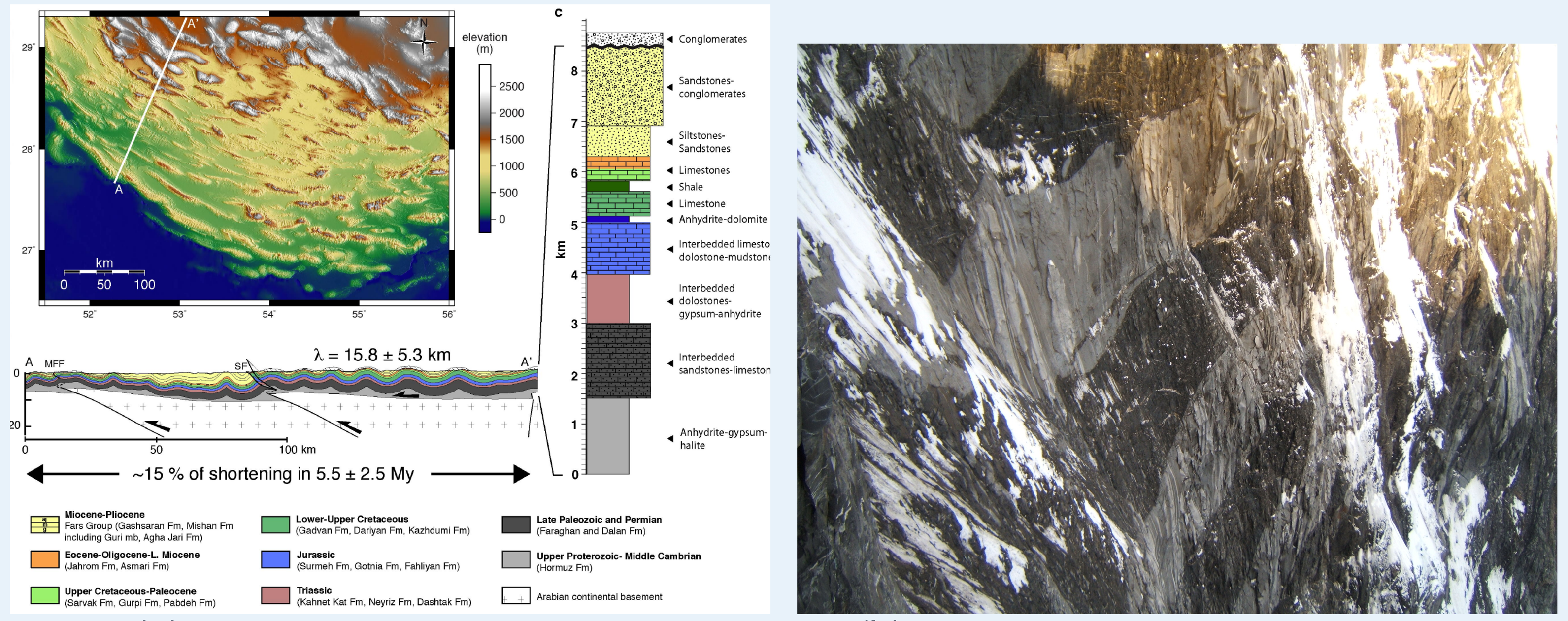
Figure: Convergence of residual norm.



(a) It 65 (b) It 74 (c) It 115

Figure: Convergence of heterogeneous $\mathfrak{p} = 1.3$, $\gamma_0 \in [10^{-2}, 1]$ with $q = 0.2$ friction at right boundary.

Heterogeneous media: the bane of adaptive mesh refinement



(a) Zagros Mtns [Yamato et al (2011)] (b) Layered granite and diorite on Mt Moffit

Figure: Geology is complex at all scales

- Adaptive spatial discretizations coarsen where acceptable accuracy can be achieved on coarse grids.
- Heterogeneous media requires high resolution throughout the domain.

Full Approximation Scheme and τ corrections

The Full Approximation Scheme is a naturally nonlinear multigrid algorithm that allows flexible incorporation of multilevel information.

- classical formulation: "coarse grid *accelerates* fine grid solution"
- τ formulation: "fine grid improves accuracy of coarse grid"
- To solve $Nu = f$, recursively apply

$$\text{pre-smooth} \quad \tilde u^h \leftarrow S^h_{\text{pre}}(u^h_0, f^h)$$

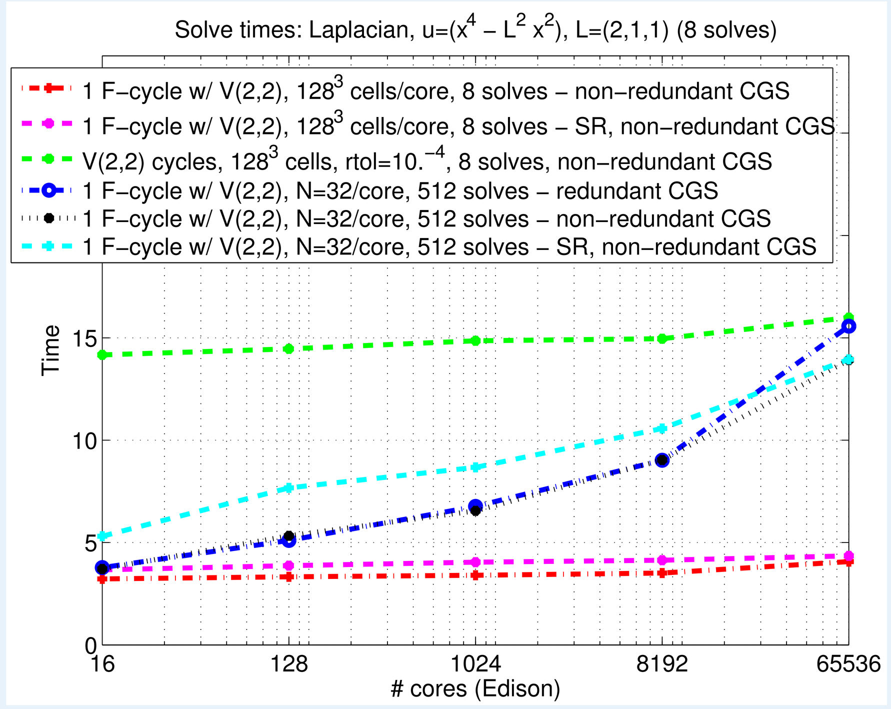
$$\text{solve coarse problem for } u^H \quad N^H u^H = \underbrace{I^H_h f^h}_{f^H} + \underbrace{N^H \hat I^H_h \tilde u^h - I^H_h N^h \tilde u^h}_{\tau^H_h}$$

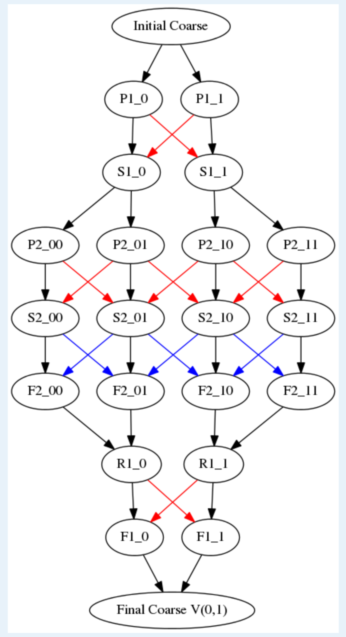
$$\text{correction and post-smooth} \quad u^h \leftarrow S^h_{\text{post}}\left(\tilde u^h + I^h_H(u^H - \hat I^H_h \tilde u^h), f^h\right)$$

I^H_h	residual restriction	$\hat I^H_h$	solution restriction
I^h_H	solution interpolation	$f^H = I^H_h f^h$	restricted forcing
$\{S^h_{\text{pre}}, S^h_{\text{post}}\}$	smoothing operations on the fine grid		

- At convergence, $u^{H*} = \hat I^H_h u^{h*}$ solves the τ-corrected coarse grid equation $N^H u^H = f^H + \tau^H_h$, thus τ^H_h is the "fine grid feedback" that makes the coarse grid equation accurate.

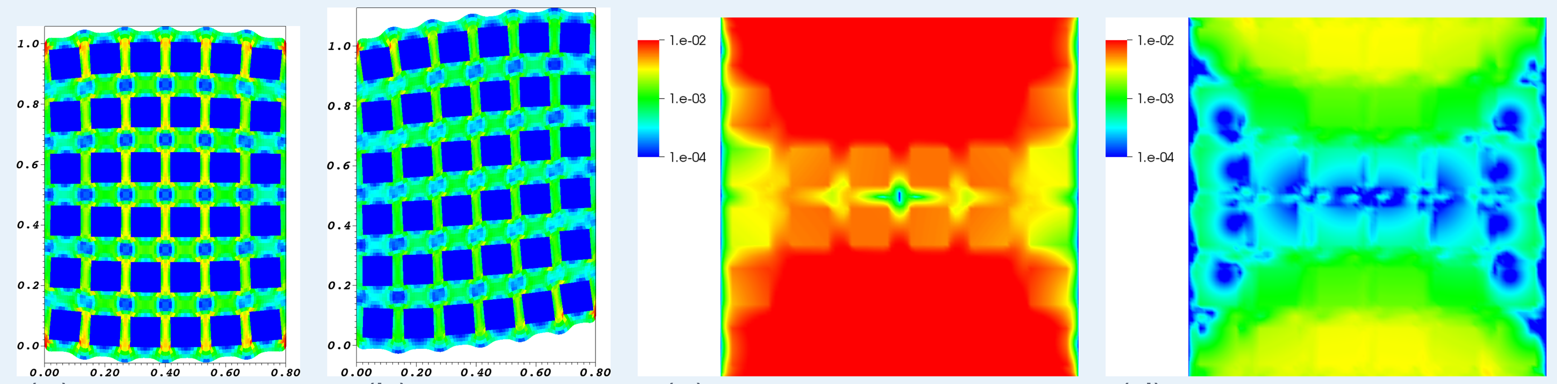
Removing data dependencies with Segmental Refinement





Introduce overlap to avoid horizontal communication in fine-grid visits. [1]

τ-adaptivity



(a) Initial solution (b) Increment (c) Smoothed error (no τ) (d) Smoothed error (with τ)

Figure: Heterogeneous strain test using 2-level multigrid with coarsening factor of 3^2. The coarse (respectively fine) grid has 3 (9) Q_1 elements across each block and 2 (6) elements across each gap. Panes (a) and (b) show the deformed body colored by strain. The initial problem of compression by 0.2 from the right is solved (a) and $\tau = A^H \hat I^H_h u^h - I^H_h A^h u^h$ is computed. Then a shear increment of 0.1 in the y direction is added to the boundary condition, and the coarse-level problem is resolved, interpolated to the fine-grid, and a post-smoother is applied. When the coarse problem is solved without a τ correction (c), the displacement error is nearly $10\times$ larger than when τ is included in the right hand side of the coarse problem (d).

Only visit fine grid where τ is "stale".

Comparison to nonlinear domain decomposition

- ASPIN (Additive Schwarz preconditioned inexact Newton) [2]
  - More local iterations in strongly nonlinear regions
  - Each nonlinear iteration only propagates information locally
  - Many real nonlinearities are activated by long-range forces
    - faults, friction, locking in granular media
  - Two-stage algorithm has different load balancing
    - Nonlinear subdomain solves
    - Global linear solve
- τ adaptivity
  - Minimum effort to communicate long-range information
  - Nonlinearity sees effects as accurate as with global fine-grid feedback
  - Fine-grid work always proportional to "interesting" changes

Status

- Running proof of concept experiments
- Library implementation underway
- Need dynamic load balancing
- Need locally computable estimates for refreshing τ
- Robust local coarsening, perhaps GenEO [3, 4]

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